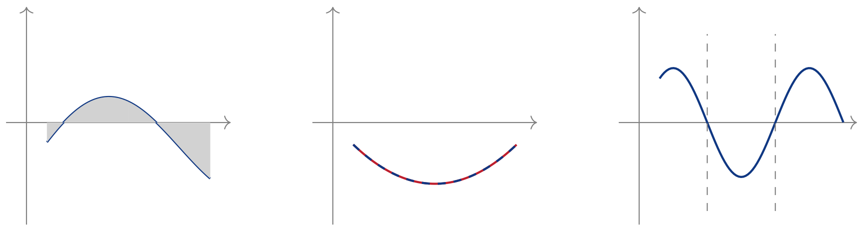

Net Signed Area **King's Rule Reflection** **Periodic Tiles**

Definite Integration Capsule

Comprehensive Formula Reference for IIT JEE Advanced

Modules: Foundation & Direct Calculus • Symmetry & King/Queen Toolbox • Periodicity & Infinite Sums • Leibniz, Estimation & Reduction

Contents

5-SEC CHECK: Net Area? | Odd/Even Zero? | King's $(a+b-x)$? | Periodic Tile? | Piecewise/GIF? | Leibniz? | Wallis/Recurrence?

MODULE 1: FOUNDATION, DIRECT CALCULUS & ELEMENTARY TRANSFORMS

0.1 1. Definite Integral as Net Signed Area & Root Existence Theorem

WHEN IT CLICKS:

Curve profile is geometric, or integral value equals 0 to detect roots.

$$\int_a^b f(x)\,dx = (\text{Area above } x\text{-axis}) - (\text{Area below } x\text{-axis})$$

Limit Swap: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Root Theorem: If $f(x)$ is continuous on $[a,b]$ and $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ has at least one root in (a,b).

Intuition

Algebraic net area. A zero net integral of a continuous curve guarantees at least one x-axis crossing.

Q: If $\frac{C_1}{1} + \frac{C_2}{1} + \frac{C_3}{1} = 0$, prove $C_2x^2 + C_1x + C_0 = 0$ has at least one root in $(0,1)$.

App: Notice

$$\int_0^1 (C_2x^2 + C_1x + C_0)\,dx = \left[\frac{C_2x^3}{3} + \frac{C_1x^2}{2} + C_0x\right]_0^1 = \frac{C_2}{3} + \frac{C_1}{2} + C_0 = 0 \Rightarrow \text{root exists in } (0,1).$$

0.2 2. Core Properties: Dummy Variable, Limit Reversal & Interval Additivity

WHEN IT CLICKS:

Given discrete integral values, inverted boundaries, or a chain of consecutive limits.

(P-1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(P-2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(P-3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Intuition

Variable of integration is dummy. Additivity connects segmented subintervals into one complete path.

Q: If $f'(x) = \frac{e^{\sin x}}{x}$ $(x > 0)$ and $\int_1^4 \frac{2e^{\sin(x^2)}}{x}\,dx = f(k) - f(1)$, find k.

App: Multiply num/den by x:

$$I = \int_1^4 \frac{e^{\sin(x^2)}}{x^2} \cdot 2x\,dx \xrightarrow{x^2=t} \int_1^{16} \frac{e^{\sin t}}{t}\,dt = \int_1^{16} f'(t)\,dt = f(16) - f(1) \Rightarrow k = 16.$$

0.3 3. Direct FCT & Antiderivatives Across Piecewise Partitions

WHEN IT CLICKS:

The integrand changes analytic definition at internal partition boundaries.

Intuition

Never apply a single antiderivative formula across a transition point. Integrate each continuous branch independently.

Q: If $f(x) = x$ $(x < 1)$ and $\sin x$ $(x \geq 1)$, evaluate $\int_0^\pi f(x)\,dx$.

App: Split at $x = 1$:

$$I = \int_0^1 x\,dx + \int_1^\pi \sin x\,dx = \left[\frac{x^2}{2}\right]_0^1 + [-\cos x]_1^\pi = \frac{1}{2} + (-(-1) + \cos 1) = \frac{3}{2} + \cos 1.$$

0.4 4. First Principle: Definite Integral as Limit of a Sum

WHEN IT CLICKS:

Question asks to evaluate $\int_a^b f(x)\,dx$ via first principles or as an explicit n-strip sum.

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a + rh) = (b-a) \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(a + r\frac{b-a}{n}\right), \quad h = \frac{b-a}{n}$$

Intuition

Total area equals sum of n infinitesimal rectangles. Apply standard polynomial sum formulas ($\sum r,\ \sum r^2$).

Q: Evaluate $\displaystyle\int_0^2 (x^2+1)\,dx$ using first principles (limit of a sum).

App: $a=0,\ b=2,\ h=\dfrac{2}{n}$:

$$I = 2\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\left[\left(\frac{2r}{n}\right)^2+1\right] = 2\lim_{n\to\infty}\left[1+\frac{4}{n^3}\cdot\frac{n(n-1)(2n-1)}{6}\right] = 2\left(1+\frac{4}{3}\right) = \frac{14}{3}.$$

0.5 5. Integration by Substitution & Simultaneous Limit Transformation

WHEN IT CLICKS:

Integrand contains a composite function $g(x)$ paired with its differential $g'(x)\,dx$.

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$$

Limits: $x=a \Rightarrow t=g(a), \quad x=b \Rightarrow t=g(b)$

Intuition

Never back-substitute to x in definite integration; evaluate fully in t using the transformed boundaries.

Q: Evaluate $\displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ where $\beta>\alpha$.

App: Put $x=\alpha\cos^2\theta+\beta\sin^2\theta \Rightarrow dx = 2(\beta-\alpha)\sin\theta\cos\theta\,d\theta$. Limits: $\theta\in[0,\frac{\pi}{2}]$:

$$I = \int_0^{\pi/2} 2\,d\theta = \pi.$$

0.6 6. Wallis Formula (Single & Dual Trig Powers on $[0,\pi/2]$)

WHEN IT CLICKS:

Integrand contains pure powers $\sin^n x$, $\cos^n x$ or products $\sin^m x\cos^n x$ over $[0,\pi/2]$.

(a)

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\cdots}{n(n-2)(n-4)\cdots}\,K, \quad K = \begin{cases} \pi/2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

(b)

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)(m+n-4)\cdots}\,K, \quad K = \begin{cases} \pi/2 & \text{both } m,n \text{ even} \\ 1 & \text{otherwise} \end{cases}$$

Intuition

Count down in steps of 2 in numerator and denominator until reaching 1 or 2. Multiply by $\pi/2$ only when all powers are even!

Q: Evaluate $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\,dx$.

App: Integrand is even $\Rightarrow$

$$2\int_0^{\pi/2} \sin^4 x \cos^6 x\,dx = 2 \cdot \frac{(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi}{256}.$$

MODULE 2: SYMMETRY, REFLECTION & THE KING / QUEEN TOOLBOX

0.7 7. Even/Odd Parity Test on Symmetric Limits $[-a, a]$ (P-4)

WHEN IT CLICKS:

Integration limits are symmetric: $[-a, a]$. Always test $f(-x)$ before doing algebra!

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f(-x) = -f(x) \text{ (odd)} \\ 2\displaystyle\int_{0}^{a} f(x)\,dx & f(-x) = f(x) \text{ (even)} \end{cases}$$

Intuition

Odd functions cancel identically across the origin. If integrand is odd, the answer is 0 immediately.

Q: Evaluate $\displaystyle\int_{-1/2}^{1/2} \cos x \cdot \ln\left(\frac{1+x}{1-x}\right) dx$.

App:

$$f(-x) = \cos(-x)\ln\left(\frac{1-x}{1+x}\right) = \cos x\left[-\ln\left(\frac{1+x}{1-x}\right)\right] = -f(x).$$

Strictly odd on $[-\frac{1}{2}, \frac{1}{2}] \Rightarrow I = 0$.

0.8 8. Parity of Composite Products, Determinants & Higher Derivatives

WHEN IT CLICKS:

Integrand has a determinant or $f(x), f'(x), f''(x)$ with known parity of f on $[-a, a]$.

Even $\times$ Even = Even | Odd $\times$ Odd = Even | Odd $\times$ Even = Odd

Derivatives: If $f(x)$ is odd $\Rightarrow f'(x)$ is even $\Rightarrow f''(x)$ is odd.

Intuition

Evaluate composite parity directly without expanding.

Q: If $f(x)$ is an odd determinant, find $\int_{-\pi/2}^{\pi/2}(x^2+1)[f(x)+f''(x)]\,dx$.

App: $f(x)$ is odd $\Rightarrow f''(x)$ is odd $\Rightarrow f(x)+f''(x)$ is odd. Multiplying by even (x^2+1) yields an odd integrand $\Rightarrow I=0$.

0.9 9. King's Property: Interval Reflection $x \to a+b-x$ (P-5)

WHEN IT CLICKS:

Integrand has complementary pairs $\left(\frac{\sin x}{\cos x}, \frac{\sqrt{x}}{\sqrt{a+b-x}}\right)$ that swap upon reflection.

$$I=\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx \Rightarrow 2I=\int_a^b [f(x)+f(a+b-x)]\,dx$$

Special case: $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

Intuition

Reflection produces a twin integral. Adding $I+I$ collapses complicated numerators/denominators to 1.

Q: Evaluate $I=\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$.

App: $a+b=5 \Rightarrow x\to 5-x$. Transformed: $I=\int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$. Adding gives $2I=\int_2^3 1\,dx=1\Rightarrow I=\frac{1}{2}$.

0.10 10. King's Multiplier Elimination Trick: Removing the "x" Factor

WHEN IT CLICKS:

An algebraic factor x is multiplied to a symmetric expression: $\int_a^b xf(x)\,dx$ where $f(a+b-x)=f(x)$.

$$\int_a^b xf(x)\,dx=\frac{a+b}{2}\int_a^b f(x)\,dx$$

For $[0,\pi]$: $\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

Intuition

Linear multiplier x prevents direct integration. King's reflection replaces x with $(a+b-x)$, canceling x upon addition.

Q: Evaluate $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$.

App: Apply King's $x \to \pi - x$:

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx \Rightarrow I = \frac{\pi}{2}[-\tan^{-1}(\cos x)]_0^{\pi} = \frac{\pi^2}{4}.$$

0.11 11. Queen's Property: Upper Limit Halving / $2a$ Rule (P-6)

WHEN IT CLICKS:

Upper limit is $2a$ (e.g. $\pi, 2\pi$) and you wish to halve the interval to a (e.g. $\pi/2$).

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Intuition

Tests mirror symmetry around midpoint $x = a$. Halves the limit and doubles the integral.

Q: Evaluate $I = \int_0^{\pi} \cot x \cos 2x\,dx$.

App: $2a = \pi$. Check $f(\pi - x) = \cot(\pi - x)\cos(2\pi - 2x) = -\cot x \cos 2x = -f(x) \Rightarrow I = 0$ immediately by Queen's rule.

0.12 12. Standard Log-Trigonometric Benchmark & Fourier Invariance

WHEN IT CLICKS:

Log-trigonometric forms appearing over domain $[0, \pi/2]$ or reducible via King's property.

$$\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx = -\frac{\pi}{2}\ln 2 \quad \Big| \quad \int_0^{\pi/2} \ln(\tan x)\,dx = 0$$

Intuition

Adding sine/cosine twins produces $\ln\left(\frac{\sin 2x}{2}\right)$. Using substitution $2x = t$ yields self-similar equation $2I = I - \frac{\pi}{2}\ln 2$.

Q: Evaluate $I = \int_0^{\pi/2} (2 \ln \sin x - \ln \sin 2x)\, dx$.

App: Expand $\ln \sin 2x = \ln 2 + \ln \sin x + \ln \cos x \Rightarrow$

$$I = \int_0^{\pi/2} [\ln \sin x - \ln \cos x - \ln 2]\, dx = 0 - \frac{\pi}{2} \ln 2 = -\frac{\pi}{2} \ln 2.$$

MODULE 3: PERIODICITY, PIECEWISE DISCONTINUITY & INFINITE SUMS

0.13 13. Periodic Functions — The Complete Toolbox (P-7 / Jack's Property)

WHEN IT CLICKS:

Integrand is periodic with fundamental period T: $f(x+T) = f(x)$ ($|\sin x| \to \pi$, $\{x\} \to 1$, $\sin^2 x \to \pi$).

(i) $\displaystyle\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$

(ii) $\displaystyle\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx$ (independent of a)

(iii) $\displaystyle\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$

(iv) $\displaystyle\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx$

(v) $\displaystyle\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx$

(vi) $\displaystyle\int_a^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx + n\int_0^T f(x)\,dx$

Intuition

Count number of fundamental periodic tiles. Shifted base bounds cancel identically across complete periods.

Q: Evaluate $I = \displaystyle\int_{-3}^{5} e^{[x]}\,dx$ where $\{x\}$ is the fractional part function.

App: $\{x\}$ has period $T = 1$. Span on $[-3, 5]$ is $5 - (-3) = 8$ complete periods $\Rightarrow I = 8\displaystyle\int_0^1 e^x\,dx = 8(e-1)$.

0.14 14. Modulus Integrands $|g(x)|$ & Zero-Crossing Sign Partitioning

WHEN IT CLICKS:

Integrand contains $|g(x)|$ or trigonometric absolute values $|\sin x|$, $|\cos x|$.

Locate roots $g(x) = 0$ inside $[a, b]$:

$$\int_a^b |g(x)|\,dx = \int_a^c -g(x)\,dx + \int_c^b g(x)\,dx \quad [g(x) \le 0 \text{ on } [a,c],\ g(x) \ge 0 \text{ on } [c,b]]$$

Intuition

Find all roots where sign flips. Strip modulus bars with appropriate $\pm$ signs for each interval.

Q: Evaluate $I = \displaystyle\int_0^{16\pi/3} |\sin x|\,dx$.

App: $|\sin x|$ has period π. Decompose:

$$I = \int_0^{5\pi} |\sin x|\,dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x|\,dx = 5\int_0^{\pi} \sin x\,dx + \int_0^{\pi/3} \sin x\,dx = 5(2) + [1 - \cos(\pi/3)] = \frac{21}{2}.$$

0.15 15. Greatest Integer $[g(x)]$ & Fractional Part $\{x\}$ Step Integrals

WHEN IT CLICKS:

Floor function $[g(x)]$ or fractional part $\{x\} = x - [x]$ appears in the integrand.

Find all points where $g(x) = k \in \mathbb{Z}$. Break $[a, b]$ into intervals where $[g(x)]$ is constant:

$$\int [g(x)]\,dx = \sum k_i \cdot \Delta x_i$$

Intuition

$[g(x)]$ is piecewise constant between integer transition points. Calculate area as sum of flat slabs.

Q: Evaluate $I = \displaystyle\int_1^2 \left(x^2 + [x^2]^2\right) dx$.

App: For $x \in [1, 2] \Rightarrow x^2 \in [1, 4]$. Transitions at $x = 1, \sqrt{2}, \sqrt{3}, 2$:

$$I = \int_1^{\sqrt{2}} (x^2 + 1^2)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^2)\,dx + \int_{\sqrt{3}}^{2} (x^2 + 3^2)\,dx.$$

0.16 16. Sum of Infinite Series via Definite Integration (Limit of a Sum)

WHEN IT CLICKS:

Infinite series limit containing terms in r/n as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{r=\phi_1(n)}^{\phi_2(n)} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_A^B f(x)\,dx$$

Replacements: $\frac{r}{n} \to x, \quad \frac{1}{n} \to dx, \quad A = \lim_{n\to\infty} \frac{\phi_1(n)}{n}, \quad B = \lim_{n\to\infty} \frac{\phi_2(n)}{n}$

Intuition

Factor out $1/n$ as differential dx, group remaining terms as $r/n = x$, determine limits from boundary indices.

Q: Evaluate $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{r}{n^2} \sec^2(r^2/n^2)$.

App: Factor $1/n$:

$$\lim \sum \frac{1}{n}\left(\frac{r}{n}\right) \sec^2\left(\frac{r^2}{n^2}\right) = \int_0^1 x \sec^2(x^2)\,dx \xrightarrow{x^2=t} \frac{1}{2}\int_0^1 \sec^2 t\,dt = \frac{\tan 1}{2}.$$

0.17 17. Series Conversion with Extended Bounds (kn terms) & Radical Forms

WHEN IT CLICKS:

Series index runs up to kn terms (e.g. $3n, 4n$) or has $\sqrt{n}, \sqrt{r}$ structures.

Rule: Upper limit is not always 1! If sum index runs to kn:

$$B = \lim_{n\to\infty} \frac{r_{\max}}{n} = \lim_{n\to\infty} \frac{kn}{n} = k \quad \text{(ensured denominator is scaled to pure } r/n\text{)}$$

Intuition

Always calculate upper limit from max index $r_{\max}/n$. Never default to integrating on $[0, 1]$ without checking.

Q: Evaluate $\lim\limits_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{4n}\right]$.

App: Write as $\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{n}\cdot\frac{1}{1+r/n}$. Lower limit $=0$, Upper $=3n/n=3$:

$$I=\int_0^3\frac{dx}{1+x}=[\ln(1+x)]_0^3=\ln 4.$$

0.18 18. Involution & Reciprocal Inversion Symmetry $(x\to 1/t)$

WHEN IT CLICKS:

Reciprocal limits $[\sin\theta,\csc\theta]$, $[a,1/a]$, or functional relation $f(1/x)+x^2f(x)=0$.

Transformation: Substitute $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}\,dt$. Boundaries invert:

$$\int_a^{1/a}\to-\int_{1/a}^{a}=\int_{1/a}^{a}$$

$$I=\int_a^{1/a}f(x)\,dx=\int_{1/a}^{a}f\left(\frac{1}{t}\right)\frac{dt}{t^2}\quad\left(\text{Jacobian factor }\frac{1}{t^2}\text{ balances internal quadratic terms}\right)$$

Intuition

Inversion pairs x with $1/x$. Combined with functional equations, produces $I=-I\Rightarrow 2I=0\Rightarrow I=0$.

Q: If $f(1/x)+x^2f(x)=0$ $(x\neq 0)$, evaluate $I=\int_{\sin\theta}^{\csc\theta}f(x)\,dx$.

App: Put $x=\frac{1}{t}$:

$$I=\int_{\csc\theta}^{\sin\theta}f\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)dt=\int_{\sin\theta}^{\csc\theta}\frac{f(1/t)}{t^2}\,dt=-\int_{\sin\theta}^{\csc\theta}f(t)\,dt=-I\Rightarrow I=0.$$

MODULE 4: LEIBNIZ DIFFERENTIATION, ESTIMATION & REDUCTION FORMULAE

0.19 19. Newton–Leibniz Theorem (Differentiating Variable Limit Integrals)

WHEN IT CLICKS:

Integral has variable bounds in x: $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$. Required for derivatives, extrema, or $0/0$ limits.

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

(Upper bound derivative × upper − lower derivative × lower)

Intuition

Do not integrate $f(t)$. The derivative instantly removes the integral sign by evaluating at the variable limits.

Q: Evaluate $\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\tan\sqrt{t}\,dt}{x^4}$.

App: Form is $\frac{0}{0}$. Apply L'Hôpital + Leibniz:

$$\lim_{x\to 0^+} \frac{\sin x\tan x\cdot 2x}{4x^2} = \frac{1}{2}\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)\left(\frac{\tan x}{x}\right) = \frac{1}{2}.$$

0.20 20. Parametric Differentiation (Feynman's Rule) & Function Recovery

WHEN IT CLICKS:

Integrand contains a secondary parameter b with stubborn logs in denominator like $\int_0^1 \frac{x^b - 1}{\ln x}\,dx$.

(a) $\dfrac{d}{db}\left[\int_a^c f(x,b)\,dx\right] = \int_a^c \dfrac{\partial f(x,b)}{\partial b}\,dx$

(b) Inverse: If $g(x) = f^{-1}(x) \Rightarrow g''(x) = -\dfrac{f''(x)}{[f'(x)]^3}$

Intuition

Differentiating w.r.t. parameter b eliminates $\ln x$ in denominator. Integrate back in parameter space.

Q: Evaluate $I(b) = \int_0^1 \dfrac{x^b - 1}{\ln x}\,dx$ for $b > -1$.

App: Differentiate w.r.t. b: $I'(b) = \int_0^1 \dfrac{x^b \ln x}{\ln x}\,dx = \int_0^1 x^b\,dx = \dfrac{1}{b+1} \Rightarrow I(b) = \ln(b+1) + C$. Since $I(0) = 0 \Rightarrow I(b) = \ln(b+1)$.

0.21 21. Estimation by Range & Monotonic Extrema ($m(b-a) \le I \le M(b-a)$)

WHEN IT CLICKS:

Integral cannot be computed analytically; question asks for strict numeric bounds on $[a, b]$.

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a), \quad \text{where } m = \min_{[a,b]} f(x),\ M = \max_{[a,b]} f(x)$$

Intuition

Find global extrema m, M on $[a, b]$ (check endpoints & $f'(x) = 0$). Multiply by interval length $(b - a)$.

Q: Prove that $4 \le \int_1^3 \sqrt{3 + x^3}\,dx \le 2\sqrt{30}$.

App: $f(x) = \sqrt{3 + x^3}$ is strictly increasing on $[1, 3] \Rightarrow m = f(1) = 2$, $M = f(3) = \sqrt{30}$. Width $b - a = 2 \Rightarrow 2(2) \le I \le 2\sqrt{30} \Rightarrow 4 \le I \le 2\sqrt{30}$.

0.22 22. Inequality Comparison & Geometric Bounding (Rectangles & Triangles)

WHEN IT CLICKS:

Tighter bounds needed than flat rectangles, or comparing functions $f(x) \le g(x)$ on $[a, b]$.

(a) $f(x) \le g(x) \Rightarrow \displaystyle\int_a^b f\,dx \le \int_a^b g\,dx$

(b) $\left|\displaystyle\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

(c) $\text{Area}_{\text{rec}} < I < \text{Area}_{\text{rec}} + \text{Area}_{\text{tri}}$

Intuition

A secant triangle enclosing the convex curve provides a much tighter upper bound than a flat circumscribed rectangle.

Q: Estimate $\displaystyle\int_0^1 e^{x^2}\,dx$ using bounding rectangle and enclosing secant triangle.

App: Lower rectangle $= 1(1) = 1$. Upper bound via triangle $=$ Base Rec $+$ Triangle $= 1 + \frac{1}{2}(1)(e-1) = \frac{e+1}{2} \Longrightarrow 1 < \displaystyle\int_0^1 e^{x^2}\,dx < \frac{e+1}{2}$.

0.23 23. Trigonometric Reduction Recurrences ($\sin^n x, \tan^n x, \sin^m x \cos^n x$)

WHEN IT CLICKS:

Integral contains an integer power index n in trigonometric functions.

(a) $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx \Rightarrow I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

(b) $I_n = \displaystyle\int_0^{\pi/4} \tan^n x\,dx \Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$

(c) $I_{m,n} = \displaystyle\int_0^{\pi/2} \sin^m x \cos^n x\,dx \Rightarrow I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

Intuition

Factor $\tan^n x = \tan^{n-2} x(\sec^2 x - 1)$. Boundary terms evaluate directly to pure rational fractions.

Q: If $I_n = \displaystyle\int_0^{\pi/4} \tan^n x\,dx$, prove $I_n + I_{n-2} = \dfrac{1}{n-1}$.

App: $I_n = \displaystyle\int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\,dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - I_{n-2} \Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$.

0.24 24. Algebraic Reduction & Polynomial Recurrence Relations

WHEN IT CLICKS:

Definite integral contains high power index n on polynomial factors $I_n = \int_0^1 (1 - x^k)^n \, dx$.

Integrate by parts: $u = (1 - x^k)^n$, $v = 1$:

$$I_n = [x(1 - x^k)^n]_0^1 + kn \int_0^1 x^k (1 - x^k)^{n-1} \, dx \Rightarrow (kn + 1)I_n = kn \, I_{n-1}$$

Intuition

Decompose x^k into $1 - (1 - x^k)$ to separate the integral into I_{n-1} and I_n, then solve for the ratio.

Q: If $I_n = \int_0^1 (1 - x^4)^n \, dx$, find the ratio I_n / I_{n-1}.

App: IBP: $I_n = 4n \int_0^1 [1 - (1 - x^4)](1 - x^4)^{n-1} \, dx = 4n(I_{n-1} - I_n) \Rightarrow (4n + 1)I_n = 4n \, I_{n-1} \Rightarrow \dfrac{I_n}{I_{n-1}} = \dfrac{4n}{4n + 1}$.

JEE Master Recall Loop

Step	Check
1.	**Symmetric Parity** $[-a, a]$ — test odd/even first, kill to 0 or double to $2\int_0^a$
2.	**King's** $(a + b - x)$ / **Queen's Halving** — reflect limits, add twins
3.	**Periodic Tiles** — count complete periods, shift base bounds
4.	**Split Modulus/GIF** — break at zeros and integer transitions
5.	**Riemann Sums** — convert $1/n \to dx$, $r/n \to x$, compute $r_{\max}/n$ as upper limit
6.	**Leibniz / Wallis** — differentiate variable-limit integrals, or use power-reduction recurrences

5-SEC CHECK (reread before finalising): Net Area? | Odd/Even Zero? | King's $(a + b - x)$? | Periodic Tile? | Piecewise/GIF? | Leibniz? | Wallis/Recurrence?